itself is a phase path representing the solution $x \equiv 0$, is a singular point. In the neighborhood of the origin the phase-paths have the form

$$U^2+x^2 = D = \text{const} \tag{7}$$

of a family of circles. By comparison of Eq. (7) with the simple harmonic motion phase-paths

$$U^2+\omega^2x^2 = \text{const} \tag{8}$$

we see that for values of C close to 1 the motion is simple harmonic with period 2π.

If we write the phase-paths Eq. (5) in polar coordinates (r, θ) we obtain

$$r^2(1+\varepsilon r^2 \sin^2 2\theta/4) = (C-1)/\varepsilon \tag{9}$$

and we see that the curves have a maximum radius $r_{max} = [(C-1)/\varepsilon]^{1/2}$ at $\theta = 0$, $\pm\pi/2$, π and a minimum radius $r_{min} = [2(C^{1/2}-1)/\varepsilon]^{1/2}$ at $\theta = \pm\pi/4$, $\pm 3\pi/4$.

The period of a particular oscillation can be written in the form

$$T = 4\int_0^{r_{max}} \frac{dx}{U} = 4\varepsilon^{1/2}\int_0^{r_{max}} \frac{dx}{[C/(1+\varepsilon x^2)-1]^{1/2}} \tag{10}$$

If we make the successive substitutions

$$z = C/(1+\varepsilon x^2)-1$$

$$\phi = \tan^{-1}(z^{1/2})$$

Eq. (10) reduces to

$$T = \frac{4}{\alpha}\int_0^{\phi_1} \frac{\cos^2\phi \, d\phi}{(1-p^2\sin^2\phi)^{1/2}} \tag{11}$$

where

$$\alpha^2 = (C-1)/C^2, \quad p^2 = C/(C-1), \quad \phi_1 = \tan^{-1}[(C-1)^{1/2}] \tag{12}$$

If we now make the substitution $p\sin\phi = \sin\mu$

$$T = 4C^{1/2}\int_0^{\pi/2} (1-p^{-2}\sin^2\mu)^{1/2} d\mu = 4C^{1/2}E(p^{-1}) \tag{13}$$

where E is a complete elliptic integral of the second kind.

Reference

[1] Evensen, D. A., "Nonlinear Vibrations of an Infinitely Long Cylindrical Shell," *AIAA Journal*, Vol. 6, No. 7, July 1968, pp. 1401–1403.

Measurement of the Error of Temperature Sensors in Flowing Gases

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Nomenclature

D = wire diameter
h = convective coefficient
k = thermal conductivity of wire
Nu = Nusselt number
Pr = Prandtl number
Re = Reynolds number
T = absolute temperature
T_s = effective black body temperature of the environment of the sensor as seen by the sensor
V = gas velocity
x = coordinate along wire axis
α = absorptivity of sensor
ε = emissivity of sensor
σ = Stefan-Boltzmann constant

Subscripts

1, 2 = smaller diameter and larger diameter wire, respectively
g = any gas
a = air

WHEN a temperature transducer is placed in a flowing gas and the gas is not in thermal equilibrium with its environment, the temperature transducer does not, in general, achieve the gas temperature. It is then desirable to relate the temperature indicated by the transducer to the actual gas temperature. Often a calculation of the transducer error is impossible because radiative environment properties are unknown.

Consider a butt-welded wire thermocouple along an isotherm in a flowing gas. It is assumed that the thermocouple junction is in a region of the wire approximately free of temperature gradients or variations of gas flow and properties. The total emissivities and absorbtivities of the thermocouple materials are assumed to be equal and constant. Catalytic reactions are assumed negligible. The wire is assumed to be a circular cylinder. The axial coordinate, x, coincides with the centerline of the wire.

The energy balance equation for the wire near the thermocouple junction is

$$(DK/4)(\partial^2 T/\partial x^2)+h(T_g-T)-\varepsilon\sigma T^4+\alpha\sigma T_s^4 = 0 \tag{1}$$

The first term of Eq. (1) can be set equal to zero near the thermocouple junction because it has been assumed that the temperature does not vary with x. Then differentiating the remaining terms with respect to wire diameter at constant gas temperature and velocity, and solving for gas temperature yields

$$T_g = T+(h+4\varepsilon\sigma T^3)[(\partial T/\partial D)_{T_g,V}]/[(\partial h/\partial D)_{T_g,V}] \tag{2}$$

The right side of Eq. (2) can be regarded as the thermocouple temperature measurement, T, plus the correction term. Significantly, Eq. (2) does not explicitly contain T_s, a potentially complicated function of the radiative environment, because the fourth term of Eq. (1) is independent of wire diameter. This fundamental advantage of Eq. (2) over Eq. (1) in evaluating T_g applies whether or not the radiative environment is uniform on all sides. However, the two partial derivatives in the correction term of Eq. (2) must be evaluated, in addition to the convective coefficient and the emissivity, in order to evaluate the correction term. Means of evaluating the partial derivatives will now be proposed.

The partial derivative $[(\partial T/\partial D_{T_g,V})]$ can be approximated in the following way, regardless of the kind of gas, when two thermocouples with different diameters are used.

$$[(\partial T/\partial D)_{T_g,V}] \approx (T_2-T_1)/(D_2-D_1) \tag{3}$$

Temperature measurements with the thermocouples yield T_1 and T_2. The measurement of D_1 and D_2 can be accomplished with a micrometer. The partial derivative $[(\partial h/\partial D)_{T_g,V}]$ can be evaluated with the aid of the relation among the Nusselt, Prandtl, and Reynolds numbers which is valid for the particular gas and sensor shape.

As an example, for circular cylindrical wires in air at Reynolds numbers between 1 and 1000, the McAdams relation[1] is available

$$(Nu)(Pr)^{-0.3} = 0.35+0.56(Re)^{0.52} \tag{4}$$

The expression for $[(\partial h/\partial D)_{T_g,V}]$ can be derived directly from Eq. (4) but a simplification is often possible. Suppose

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$$0.56(Re)^{0.52} \gg 0.35 \quad (5)$$

Then by omitting the 0.35 term from Eq. (4) and differentiating with respect to D

$$[(\partial h/\partial D)_{T_g,V}] \approx -0.48h/D \quad (6)$$

If the inequality (5) holds, approximations (3) and (6) can be substituted in Eq. (2) and the following equation for air temperatures is obtained

$$T_a \approx T-(h+4\varepsilon\sigma T^3)(T_1-T_2)(D)/[0.48(D_1-D_2)h] \quad (7)$$

The approximation (3) is true at only one intermediate wire diameter greater than D_1 but smaller than D_2. The factors, T, D and h in expression (7) should be evaluated for the same unknown intermediate diameter. But because the required intermediate diameter is unknown the average temperature, diameter, and convective coefficient of the two thermocouples will be used instead. Expression (7) then becomes

$$T_a \approx \langle T\rangle-(\langle h\rangle+4\varepsilon\sigma\langle T\rangle^3)(T_2-T_1)(\langle D\rangle)/[0.48\langle h\rangle(D_2-D_1)] \quad (8)$$

where

$\langle T\rangle \equiv (T_1+T_2)/2$, $\langle D\rangle \equiv (D_1+D_2)/2$, and $\langle h\rangle \equiv (h_1+h_2)/2$

To minimize the error caused by using the average values of T, D and h in expression (8), the difference between D_2 and D_1 should be as small as is practical.

A very useful simplification of expression (8) is possible when

$$\langle h\rangle \gg 4\varepsilon\sigma\langle T\rangle^3 \quad (9)$$

Expression (8) then reduces to

$$T_a \approx \langle T\rangle-(T_1-T_2)\langle D\rangle/[0.48(D_1-D_2)] \quad (10)$$

When inequality (9) applies, the correction term in expression (10) is very easy to obtain because it is a function of only the two thermocouple temperatures and the two diameters. Each of these is easily measured.

An experimental investigation of expression (10) was conducted for two Chromel-Alumel thermocouples. They were butt-welded and each resembled one continuous wire. The wire diameters were 0.0099 in. and 0.0159 in. The thermocouples were consecutively placed at each of three different levels at the exit plane of a horizontal air duct (see Fig. 1). The lowest level was $\frac{3}{4}$ in. above the top of a 600-w hot plate which provided a source of thermal radiation for the wires but could not heat the flowing air in the duct upstream of the wires. The thermocouple junction was placed at the middle of the segment of the wire exposed to the flowing air. The air flow rate was 1.32 lb/min. The inside diameter of the duct was 1.85 in.

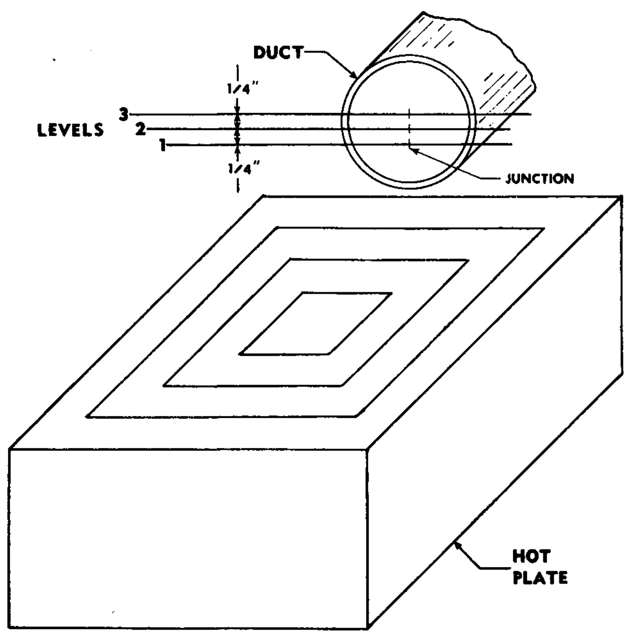

Fig. 1 Test configuration.

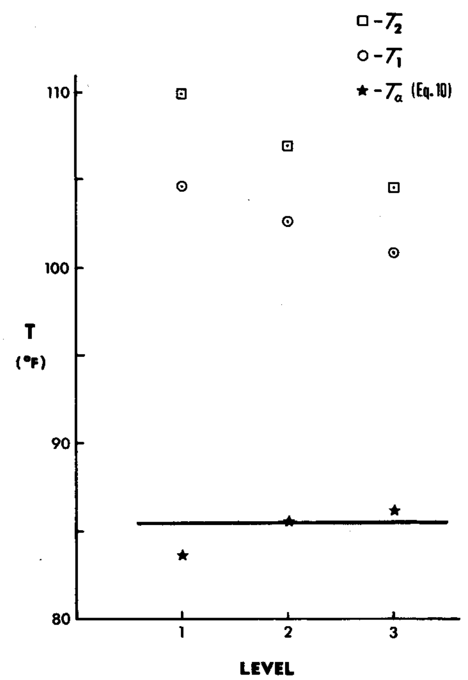

Fig. 2 Experimental results.

Thermocouple data were taken with a Leeds and Northrop type K-3 universal potentiometer. The temperature of the flowing air was measured to be 85.4°F before and after the experiment with the hot plate removed. The air temperature was close enough to ambient temperature to eliminate the necessity of correction of measurements when the hot plate was removed. Each thermocouple material was from a single batch and it was verified that the thermocouples gave equal outputs at equal temperatures. The data T_1 and T_2 taken when the hot plate was radiating, are shown in Fig. 2. The solid line in Fig. 2 represents the true flowing air temperature, 85.4°F.

Using the average air velocity in the duct, the Reynolds number for the smaller thermocouple was calculated to be approximately 79. The left side of inequality (5) is thus greater than the right side by a factor of approximately 15. Therefore, approximation (6) is justified. Also the left side of inequality (9) is approximately fifty times larger than the right side, even if the emissivity equals unity. Thus expression (10) was used to calculate the air temperatures. The results are also shown in Fig. 2.

Some scatter in the three calculated values for air temperature was caused by vertical position errors. The wires could be set only to within $\pm\frac{1}{64}$ in. of their intended levels. It can be deduced from the data in Fig. 2 that in the worst case the position error would introduce an error of approximately 10% in the correction term of exression (10). The calculated air temperature at level one was in error -1.8°F, or about -8% of the correction term. At levels two and three, the absolute error in the correction term was smaller.

Expression (8) reveals that when the temperature of a hot gas is sought in the presence of a cold environment, an artificial radiative heat source could be used to raise the temperature of each wire. Eventually the wire temperatures would be equal in the same steady radiative environment. At that temperature the correction term equals zero and the thermocouple temperatures equal the gas temperature.

It is concluded that the thermocouple error caused by forced convection and radiation in gas temperature measurements can be determined experimentally. This is possible if two thermocouples of different diameters are used consecutively at the location at which the gas temperature is desired. The gas temperature must be steady and the thermocouples must be placed along an isotherm in the gas. The thermocouple junction must be at a location in the wire where the second derivative of temperature with respect to the axial coordinate of the thermocouple wire is negligible. No information about the radiative environment is required but the radiative and convective heat-transfer characteristics of the sensor must be known in the general case. Evaluation of a geometric shape-emissivity factor[2] is never required.

The principle of operation can be extended to any kind of temperature sensor if two geometrically similar sensors of slightly different sizes are available and if there is no conduction loss through the sensor supports.

References

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Smooth Empirical Bayes Estimation of Observation Error Variances in Linear Systems

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Introduction

In Ref. 1, an empirical Bayes estimator was developed for estimating the unknown random observation error variances in a discrete time linear system. There it was assumed that each unknown variance could be represented as the product of a known nominal value and an unknown random scale factor which is to be estimated.

A continuous empirical Bayes smoothing technique is developed in Ref. 2. This technique provides estimators possessing smaller average squared error losses than the type of empirical Bayes estimator employed in Ref. 1. A similar smooth empirical Bayes estimator was developed in Ref. 3, where a continuous prior density function approximation was "smoothed" through a suitable function of the observation data.

In this Note, a smooth empirical Bayes estimator is developed for estimating the unknown random scale component of each observation error variance. This estimator will be shown to possess a smaller average squared error loss than the estimator presented in Ref. 1.

Scale Factor Estimation

Consider the linear discrete dynamic system given by

$$x_i = \phi_i x_{i-1} + u_{i-1} \tag{1}$$

augmented by the linear observation—state equation

$$y_i = H_i x_i + v_i, \qquad i = 1, 2, \ldots \tag{2}$$

with the same assumptions as in Ref. 1. As in Ref. 1 the observation error covariance matrix R_i is represented by the diagonal matrix

$$R_i = \operatorname{diag}(r_{i1}^2/\theta_{i1}, r_{i2}^2/\theta_{i2}, \ldots, r_{iq}^2/\theta_{iq}) \tag{3}$$

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where q is the number of observation types available, r_{ij}^2 is a known nominal value of the observation error variance associated with the jth observation type at time epoch i. Also, the scale factors $\{\theta_{ij}: i = 1, 2, \ldots\}$ are independent realizations of a random variable Θ_j having a completely unknown and unspecified prior density function $g_j(\theta)$ which is zero on the negative real numbers and which may be different for each observation type.

From the usual Gaussian assumption on v_i in Eq. (2), it follows that

$$z_{ij} = (y_{ij} - h_{ij} x_i)/r_{ij}, \qquad i = 1, 2, \ldots, n_j \tag{4}$$

conditional on θ_{ij}, is distributed with probability density function given by

$$f(z_{ij} \mid \theta_{ij}) = \theta_{ij}^{1/2}(2\pi)^{-1/2} \exp[-\theta_{ij} z_{ij}^2/2] \tag{5}$$

Here h_{ij} is the jth row of H_i and n_j is the total number of observations of type j available up to the present time. For simplicity, we let $n_j = n$ and shall drop the subscript j for the remainder of this section.

The Bayes estimator for θ_n is given by

$$E(\theta_n \mid z_n) = \int_0^\infty \theta_n f(z_n \mid \theta_n) g(\theta_n)\, d\theta_n \Big/ \int_0^\infty f(z_n \mid \theta_n) g(\theta_n)\, d\theta_n \tag{6}$$

According to a technique developed in Ref. 4 and used in Refs. 2 and 3, the prior density function $g(\theta_n)$ may be estimated by means of the approximation given by

$$g_n(\theta_n) = \frac{K}{nh(2\pi)^{1/2}} \sum_{i=1}^{n} \exp\left[-\frac{1}{2}\left(\frac{\theta_n - \hat{\theta}_i}{h}\right)^2\right], \quad 0 < \theta_n < \infty \tag{7}$$

where $h = h(n) = n^{-1/5}$, $\hat{\theta}_i$ is a suitably chosen estimate of θ_i to be discussed later, and where

$$K = \left[1 - \frac{1}{n}\sum_{i=1}^{n} \phi\left(\frac{-\hat{\theta}_i}{h}\right)\right]^{-1} \tag{8}$$

In Eq. (8), $\phi(\cdot)$ denotes the standard normal cumulative distribution function.

Inserting Eqs. (5) and (7) into Eq. (6), collecting terms, performing the indicated integrations, and simplifying, yields the smooth empirical Bayes estimator for θ_n given by

$$\tilde{\theta}_n = E_n(\theta_n \mid z_n) = \left(\frac{3}{2}h\right) \sum_{i=1}^{n} \exp[-a_i] U(2, b_i) \Big/ \sum_{i=1}^{n} \exp[-a_i] U(1, b_i) \tag{9}$$

where

$$a_i = (3z_n^4 h^4 + 8z_n^2 \hat{\theta}_i^2 h^2 + 4\hat{\theta}_i^2 - 4h^3 z_n^2 \hat{\theta}_i)/16h^2 \tag{10}$$

$$b_i = (z_n^2 h^2 - 2\hat{\theta}_i)/2h \tag{11}$$

and where $U(c, x)$ is the parabolic cylinder function[5,6] defined by

$$U(c, x) = [e^{-x^2/4}/\Gamma(c + 1/2)] \int_0^\infty e^{-xy - y^2/2} y^{c-1/2}\, dy \tag{12}$$

Numerous asymptotic expansions exist for evaluating $U(c, x)$. In particular the form given in Ref. 5 in Sec. 19.12.3, in conjunction with the asymptotic expansion in Sec. 13.1.2, was used here for evaluating Eq. (12).

The entire preceding development was undertaken on the assumption that the true state vector x_n is known. Specifically x_n was used in obtaining z_n in Eq. (4). This is clearly not the case, and the estimate for x_n given by

$$\bar{x}_n = \phi_n \hat{x}_{n-1} \tag{13}$$

where $\hat{x}_{n-1}$ is the Kalman, Empirical Bayes filter estimate for x_{n-1} given in Ref. 1, is used.

Now consider the estimates $\hat{\theta}_i$ required in Eq. (9). The preceding estimates obtained from Eq. (9) for $i = 1, 2, \ldots, n-1$, i.e. $\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_{n-1}$, can be substituted for $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{n-1}$ respectively when calculating $\tilde{\theta}_n$. Also, the maximum likelihood estimate for θ_n given by $1/z_n^2$ may be substituted for $\hat{\theta}_n$ in Eq. (9) or an estimate of θ_n such as that provided by Ref. 1 could be employed. For simplicity the maximum likelihood estimate was used here